On the outerplanar crossing numbers of complete multipartite graphs

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Abstract

We calculate the outerplanar crossing numbers of complete multipartite graphs which have n partite sets with m vertices and one partite set with p vertices, where either $p|mn$ or $mn|p$.

1 Introductory Material

An *outerplanar drawing* of a graph G is a drawing of G in which the vertices are placed on a circle and the edges are drawn as straight lines cutting through the interior. We require that in such drawings, no more than two edges cross in a single point. The *outerplanar crossing number* of a graph G is the minimum number of crossings taken over all outerplanar drawings of G. We denote the outerplanar crossing number of G by $\nu_1(G)$. If D is an outerplanar drawing of a graph G, the number of crossings in D is denoted by $cr_1(D)$. The outerplanar crossing number of a graph was defined in [Kainen 1990]. There are very few exact results known. In fact all of them can be found in two papers: [Fulek *et al.* 2005] and [Riskin 2003]. Calculating these values seems to be of interest to the VLSI community, and it is interesting also to graph theorists, and thus we offer some new results here. Note that [Shahrokhi *et al.* 1996] contains a useful introduction to the outerplanar crossing number problem, as well as some interesting lower bounds. The complete n-partite graph with each partite set containing m vertices is denoted by $K_{m^{(n)}}$. The complete multipartite graph $K_{\underbrace{a,\ldots,a}_{m},\underbrace{b,\ldots,b}_{n}}$ is denoted by $K\left(a^{(m)}, b^{(n)}\right)$. Our main results here are:

Theorem 1.1. *If $p|mn$ then*

$$\begin{aligned}\nu_1(K(p^{(1)},m^{(n)})) =& \frac{1}{24}m^4n^4+\frac{1}{12}m^2n^3-\frac{1}{12}m^4n^3-\frac{1}{4}m^3n^3+\frac{1}{2}m^3n^2+\frac{1}{24}m^4n^2-\frac{1}{4}m^3n\\ &+\frac{1}{6}m^2n^2p^2-\frac{1}{4}p^2mn-\frac{3}{4}m^2n^2p+\frac{1}{12}mnp+\frac{1}{2}m^2np-\frac{1}{6}m^3n^2p\\ &+\frac{1}{6}m^3n^3p+\frac{1}{6}mn^2p\end{aligned}$$

and

Theorem 1.2. *If $mn|p$ then*

$$\nu_1(K(p^{(1)},m^{(n)})) = \phi(m,n,p)-\frac{1}{12}m^2n^2+\frac{1}{12}p^2$$

where $\phi(m,n,p)$ is the expression for $\nu_1(K(p^{(1)},m^{(n)}))$ given in Theorem 1.1.

We will need the following fact from [Riskin 2003]:

Lemma 1.1. *If $m|n$ then $\nu_1(K_{m,n}) = \frac{1}{12}n(m-1)(2mn-3m-n)$ and this minimum value is attained when the m vertices are distributed evenly amongst the n vertices.*

Also the following, the proof of which is a mere calculation:

Lemma 1.2. *$K\left(a^{(m)},b^{(n)}\right)$ has $ma+nb$ vertices and the number of edges is given by:*

$$\frac{1}{2}ma((m-1)a+nb)+\frac{1}{2}nb((n-1)b+ma)$$

And finally the following from [Fulek *et al.*]:

Theorem 1.3. $\nu_1(K_{m^{(n)}}) = \frac{1}{24}m^2n(n-1)(m^2n^2+2n-m^2n-6mn+6m)$

2 Results

We will need the following obvious fact, the statement of which is practically the proof:

Lemma 2.1.

$$\sum_{k=1}^{mn}\left\lfloor\frac{k-1}{n}\right\rfloor = \sum_{i=0}^{m-1}\sum_{k=in+1}^{(i+1)n} i = \frac{1}{2}mn(m-1)$$

And we now prove our main theorems:

Proof of Theorems 1.1 and 1.2. Let D be an outerplanar drawing of $K(p^{(1)}, m^{(n)})$ with $n \geq 1$. Denote the p vertices of the first partite set by v_ℓ, $1 \leq \ell \leq p$. There are three kinds of crossings in D: First there are crossings in the drawing of $K_{m^{(n)}}$ induced by removing $v_1, \ldots, v_p$ from D. Second, there are crossings wholly in the isomorph of $K_{p,mn}$ induced by the set of all edges joining vertices in $\{v_1, \ldots, v_p\}$ to other vertices. Call the number of such crossings C_2. Finally there are crossings determined by one edge in the induced $K_{m^{(n)}}$ and one edge in the induced $K_{p,mn}$. Call the number of such crossings C_3. Thus

$$cr_1(D) \geq \nu_1\left(K_{m^{(n)}}\right) + C_2 + C_3$$

Let $u_{k\ell}$ be the vertex of $K_{m^{(n)}}$ which lies k spaces counterclockwise around the circle from v_ℓ. Let $e_{k\ell}$ be the edge of D joining these two vertices. Then the $k-1$ vertices of $K_{m^{(n)}}$ between $u_{1\ell}$ and $u_{(k-1)\ell}$ inclusive induce a complete multipartite graph, which we call $L_{k\ell}$. The same holds for the $mn-k$ vertices of $K_{m^{(n)}}$ between $u_{(k+1)\ell}$ and $u_{(mn)\ell}$ inclusive, and we refer to that complete multipartite graph as $R_{k\ell}$. Let $cr_1(e_{k\ell})$ denote the number of edges of the induced $K_{m^{(n)}}$ which cross $e_{k\ell}$. Note that

$$C_3 = \sum_{\ell=1}^{p} \sum_{k=1}^{mn} cr_1(e_{k\ell})$$

The edges of $K_{m^{(n)}}$ which cross $e_{k\ell}$ consist of all edges of $K_{m^{(n)}}$ which are not in $L_{k\ell}$, not in $R_{k\ell}$, and not incident to $u_{k\ell}$. The number of edges incident to $u_{k\ell}$ in $K_{m^{(n)}}$ is $m(n-1)$. Hence we can obtain a lower bound on $cr_1(e_{k\ell})$ by maximizing the number of edges in $L_{k\ell}$ and in $R_{k\ell}$. The number of edges in a complete multipartite graph with a fixed number of vertices is largest when the number of partite sets is as large as possible and the vertices are as evenly distributed among the partite sets as possible. Let r be the remainder when $k-1$ is divided by n. I.e. $r = k-1-n\left\lfloor \frac{k-1}{n} \right\rfloor$. The number of partite sets in $L_{k\ell}$ is as large as possible and the vertices are as evenly distributed as possible amongst them when r of them have $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ vertices and $n-r$ have $\left\lfloor \frac{k-1}{n} \right\rfloor$ vertices. In other words, when

$$L_{k\ell} \cong K\left(\left(\left\lfloor \frac{k-1}{n} \right\rfloor + 1\right)^{\left(k-1-n\left\lfloor \frac{k-1}{n} \right\rfloor\right)}, \left\lfloor \frac{k-1}{n} \right\rfloor^{\left(n-k+1+n\left\lfloor \frac{k-1}{n} \right\rfloor\right)}\right)$$

Note that this holds even when $1 \leq k-1 \leq n$ by interpreting $K\left(1^{(r)}, 0^{(n-r)}\right)$ in the natural way. A similar argument yields the fact that the number of edges in $R_{k\ell}$ is maximized when

$$R_{k\ell} \cong K\left(\left(\left\lfloor \frac{mn-k}{n} \right\rfloor + 1\right)^{\left(mn-k-n\left\lfloor \frac{mn-k}{n} \right\rfloor\right)}, \left\lfloor \frac{mn-k}{n} \right\rfloor^{\left(n-mn+k+n\left\lfloor \frac{mn-k}{n} \right\rfloor\right)}\right)$$

Then using Lemma 1.2 with $L_{k\ell}$ and $R_{k\ell}$, we find maximum values $M_L(k)$ and $M_R(k)$ for $E(L_{k\ell})$ and $E(R_{k\ell})$ respectively. Hence:

$$cr_1(e_{k\ell}) \geq \frac{1}{2}m^2 n(n-1) - m(n-1) - M_L(k) - M_R(k))$$

and therefore

$$\begin{aligned} C_3 &\geq \sum_{\ell=1}^{p}\sum_{k=1}^{mn}\left[\frac{1}{2}m^2n(n-1)-m(n-1)-M_L(k)-M_R(k)\right] \\ &= p\sum_{k=1}^{mn}\left[\frac{1}{2}m^2n(n-1)-m(n-1)-M_L(k)-M_R(k)\right] \end{aligned}$$

Invoking Theorem 1.1 we find that

$$C_2 \geq \nu_1(K_{p,mn}) = \begin{cases} \frac{1}{12}mn(p-1)(2pmn-3p-mn) & p|mn \\ \\ \frac{1}{12}p(mn-1)(2pmn-3mn-p) & mn|p \end{cases}$$

and hence

$$\begin{aligned} \nu_1(K(p^{(1)},m^{(n)})) \geq \nu_1(K_{m^{(n)}}) &+ \frac{1}{12}mn(p-1)(2pmn-3p-mn) \\ &+ p\sum_{k=1}^{mn}\left[\frac{1}{2}m^2n(n-1)-m(n-1)-M_L(k)-M_R(k)\right] \end{aligned} \tag{2.1}$$

if $p|mn$ and

$$\begin{aligned} \nu_1(K(p^{(1)},m^{(n)})) \geq \nu_1(K_{m^{(n)}}) &+ \frac{1}{12}p(mn-1)(2pmn-3mn-p) \\ &+ p\sum_{k=1}^{mn}\left[\frac{1}{2}m^2n(n-1)-m(n-1)-M_L(k)-M_R(k)\right] \end{aligned} \tag{2.2}$$

if $mn|p$. Applying Lemmas 1.2 and 2.1 to the expression

$$\sum_{k=1}^{mn}(M_L(k)+M_R(k))$$

and substituting back into (2.1) and (2.2) we find, after invoking Theorem 1.3, the requisite expressions. Furthermore this bound is actually attained when the vertices of each partite set are distributed evenly around the circle. Incidentally, it is interesting that if $p=m$ our proof essentially reduces to a proof by induction of Theorem 1.3 which is different from the method used in [Fulek *et al.*]. □

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Bibliography

1. Fulek, R., He, H., Sykora, O., and Vrt'o, I. Outerplanar crossing numbers of 3-row meshes, Halin graphs, and complete p-partite graphs. Lecture Notes in Computer Science 3381(2005) 376-379.

2. Kainen, P.C. The book thickness of a graph II. Congressus Numerantium 71(1990) pp. 127-32.

3. Riskin, A. On the outerplanar crossing numbers of $K_{m,n}$. Bulletin of the Institute for Combinatorics and its Applications 39(2003) pp. 7-15.

4. Shahrokhi, F., Székeley, L.A., Sýkora, O., and Vrťo, I. The book crossing number of a graph. J. Graph Theory 21(1996) pp. 413-24.